MATH2230A

Tutorial 10 16-11-2016

Topics: Laurent series and Singularity

Finding Laurent series for a given function f

1) Find the Laurent series of the function $f(z)=\frac{1}{4z-z^2}$ in the region
(i) $\{z\in\mathbb{C} \mid 0<|z|<4\}$
(ii) $\{z\in\mathbb{C} \mid |z|>4\}$

Ans: Recall that $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+\dots$ whenever $|z|<1$.

i) Since $0<|z|<4$, we have $\frac{|z|}{4}<1$.

$$f(z)=\frac{1}{4z-z^2}=\frac{1}{4z}\cdot\frac{1}{1-\left(\frac{z}{4}\right)}=\frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$$

$$\Rightarrow \quad f(z)=\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}=\frac{1}{4z}+\sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$

ii) Since $|z|>4$, we have $\frac{4}{|z|}<1$.

$$f(z)=\frac{1}{4z-z^2}=\frac{-1}{z^2}\cdot\frac{1}{1-\frac{4}{z}}=\frac{-1}{z^2}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n$$

$$\Rightarrow \quad f(z)=-\sum_{n=0}^{\infty}\frac{4^n}{z^{n+2}}$$

2) Let $f(z)=\frac{e^z}{z(z^2+1)}$. Find the following terms in the Laurent series expansion of f in the regions:
(i) first three non-zero positive degree terms in $\{z\in\mathbb{C} \mid 0<|z|<1\}$
(ii) first non-zero negative degree terms in $\{z\in\mathbb{C} \mid |z|>1\}$

Ans: Recall that $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$

i) $$f(z)=\frac{e^z}{z(z^2+1)}=\frac{1}{z}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)\left(1-z^2+z^4-z^6+\dots\right)$$

$\therefore$ degree 1 term $= \frac{1}{z}\cdot 1\cdot(-z^2) + \frac{1}{z}\cdot\left(\frac{z^2}{2!}\right)\cdot 1 = -\frac{z}{2}$

degree 2 term $= \frac{1}{z}\cdot(z)(-z^2) + \frac{1}{z}\cdot\left(\frac{z^3}{3!}\right)(1) = -\frac{5}{6}z^2$

degree 3 term $= \frac{1}{z}\cdot(1)(z^4) + \frac{1}{z}\left(\frac{z^2}{2!}\right)(-z^2) = \frac{5}{6}z^3$.

ii) $f(z) = \dfrac{e^z}{z(z^2+1)} = \dfrac{1}{z^3}\cdot e^z\cdot\dfrac{1}{1+\frac{1}{z^2}}$

$\Rightarrow f(z) = \dfrac{1}{z^3}\left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\right)\left(1 - \dfrac{1}{z^2} + \dfrac{1}{z^4} - \cdots\right)$

$\therefore$ degree (-1) term $= \frac{1}{z^3}\left(\frac{z^2}{2!}\cdot 1 + \frac{z^4}{4!}\cdot\left(\frac{-1}{z^2}\right) + \frac{z^6}{6!}\cdot\frac{1}{z^4} + \frac{z^8}{8!}\left(\frac{-1}{z^6}\right) + \cdots\right)$

$= \left(\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{(2n)!}\right)z^{-1}$.

3) Find the Laurent series of $g(z) = \frac{1}{z}\sin^2\frac{1}{2z}$ in $\{z \mid 0<|z|<+\infty\}$.

Ans: Since $\sin^2\dfrac{1}{2z} = \dfrac{1-\cos\frac{1}{z}}{2}$

$= \dfrac{1}{2}\left(1 - \left(1 - \dfrac{(\frac{1}{z})^2}{2!} + \dfrac{(\frac{1}{z})^4}{4!} - \dfrac{(\frac{1}{z})^6}{6!} + \cdots\right)\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{2!z^2} - \dfrac{1}{4!z^4} + \dfrac{1}{6!z^6} - \cdots\right)$

$= \dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n)!z^{2n}}$

$\therefore g(z) = \dfrac{1}{z}\sin^2\dfrac{1}{2z} = \dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n)!z^{2n+1}}$

4) Find the Laurent series of $f(z) = \frac{1}{1-z^2}$ in $D = \{z \mid 0 < |z-1| < 2\}$.

Ans: $f(z) = \dfrac{1}{1-z^2} = \dfrac{1}{(1-z)(1+z)} = \dfrac{-1}{(z-1)(1+z)} = \dfrac{-1}{(z-1)(2+(z-1))}$

$\Rightarrow f(z) = \dfrac{-1}{z-1}\cdot\dfrac{1}{2(1+\frac{z-1}{2})} = \dfrac{-1}{2(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}(z-1)^{n-1}}{2^{n+1}}$

5) Find the Laurent series of $f(z)=\frac{e^{2z}}{(z-i)^3}$ in $\{z \mid 0<|z-i|<+\infty\}$

Ans: $f(z)=\frac{e^{2z}}{(z-i)^3}=e^{2i}\frac{e^{2(z-i)}}{(z-i)^3}=\frac{e^{2i}}{(z-i)^3}\sum_{n=0}^{\infty}\frac{(2(z-i))^n}{n!}$.

$$\Rightarrow \quad f(z)=e^{2i}\sum_{n=0}^{\infty}\frac{2^n(z-i)^{n-3}}{n!}$$

Three types of singularity:

If f has an isolated singularity at z_0, and

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}, \quad 0<|z-z_0|<R$$

then we have three possibilities:

① $b_n=0 \ \forall n\in\mathbb{N}$.
In this case we say that f has a removable singularity at z_0.

② $b_n=0$ except finitely many n.
In this case we say that f has a pole of order N, where $N=\max\{n \mid b_n\neq 0\}$.

③ $b_n\neq 0$ for infinitely many n.
In this case we say that f has an essential singularity at z_0.

Example: Look back to the previous examples,

1) $f(z)=\frac{1}{4z-z^2}=\frac{1}{4z}+\sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}}$ has a pole of order 1 / simple pole at $z=0$.

2) $f(z)=\frac{e^z}{z(z^2+1)}=\frac{1}{z}\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1-z^2+z^4-\cdots\right)$
has a simple pole at $z=0$.

3) $g(z)=\frac{1}{z}\sin^2\frac{1}{2z}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!\,z^{2n+1}}$ has an essential singularity at $z=0$.